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Generalized Symmetries

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Snowmass white paper arXiv: 2205.09545 with
Clay Cordova, Thomas Dumitrescu, and Kenneth Intriligator
See also McGreevy arXiv:2204.03045

Symmetry

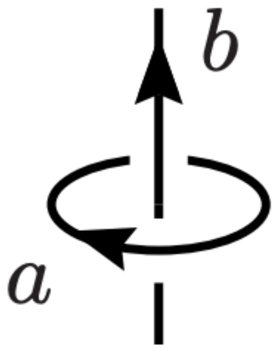
- Symmetry has proven, from time and again, to be of fundamental importance for describing Nature.
- In recent years, there has been a revolution in our understanding of global symmetries.
- The notion of global symmetry has been generalized in different directions.
- These generalized global symmetries are some of the few **universally applicable** tools to analyze general quantum systems, not limited to supersymmetric or solvable models.

Generalized global symmetries

- These new symmetries lead to several surprising consequences:
  - generalized ‘t Hooft anomaly matching conditions
  - new implications for the phase diagram of gauge theories
  - new organizing principles of topological phases in condensed matter physics
  - new insights into naturalness problems
- Active collaboration between experts from high energy physics, condensed matter physics, quantum gravity, and mathematics.
- In this talk I’ll discuss only some of these developments. Please see the white paper for more references. I apologize in advance for the variety of fascinating papers that are not discussed below.

Many other generalizations of global symmetries not discussed here, e.g. dipole symmetry, asymptotic symmetry,…

Generalizations



Higher-form symmetries

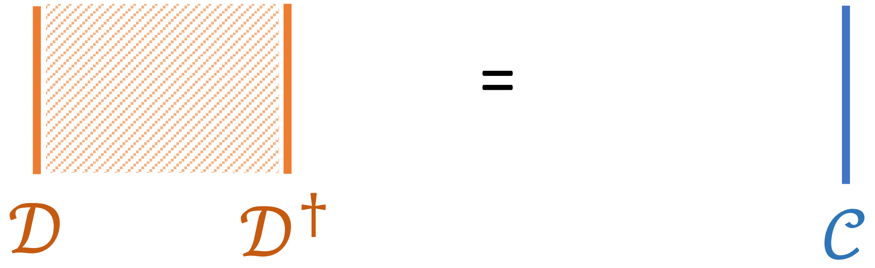
e.g. center symmetry in gauge theory

Subsystem symmetries

e.g. fractons

Non-invertible symmetries

e.g. Ising model, 4d Maxwell theory, QED, QCD…



Symmetry and topology

- Consider a QFT in d-spacetime dimensions with a conserved Noether current $\partial_\mu j^\mu = 0$. We can define a $(d-1)$-form $J_{\mu_1\cdots\mu_{d-1}} = \epsilon_{\mu_1\cdots\mu_d} j^{\mu_d}$.
- The conserved $U(1)$ symmetry operator is

$$U_\theta = \exp(i\theta \int d^{d-1}x\, j^0) = \exp(i\theta \int d^{d-1}x\, J_{1\cdots(d-1)})$$

- For a relativistic QFT, the time direction is on the same footing as any other spatial direction [Einstein 1905]. We can therefore integrate the current on a general closed $(d-1)$-manifold M in d-dimensional Euclidean spacetime:

$$U_\theta(M) = \exp(i\theta \oint_M J)$$

- The conservation equation $\partial_\tau U_\theta = 0$ is now <u>upgraded</u> to the fact that $U_\theta(M)$ depends on M only topologically (divergence theorem).

Ordinary global symmetry

Properties of symmetry op.	Ordinary symmetry $U_g\left(M^{(d-1)}\right)$	**Example**: $U(1)$ $\exp(i\theta \oint_{M^{(d-1)}} J^{(d-1)})$
Codimension in spacetime	1	$J^{(d-1)}$ is a $d-1$-form
Topological	yes	$J^{(d-1)}$ is closed, $dJ^{(d-1)} = 0$
Fusion rule	group $U_{g_1}U_{g_2} = U_{g_1 g_2}$	$U(1)$ $U_{\theta_1}U_{\theta_2} = U_{\theta_1+\theta_2}$

Next, we generalize the ordinary global symmetry by modifying the above conditions.

Generalized global symmetries

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible symmetry
Codimension in spacetime	1	>1	>1	≥ 1
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	category $\mathcal{D} \times \mathcal{D}^\dagger \neq 1$

Higher-form Symmetry

Global symmetries and generalizations

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible operator
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Higher-form symmetries and anomalies
[Gaiotto-Kapustin-Seiberg-Willett 2014,…]

- The simplest example of higher-form symmetries is the one-form center symmetry in gauge theory. E.g. $\mathbb{Z}_N$ center symmetry in $SU(N)$ Yang-Mills theory. It acts on the Wilson lines, rather than the local operators.
- Higher-form global symmetries can have anomalies, which prevent us from gauging them. These anomalies lead to generalized 't Hooft anomaly matching conditions. Nontrivial constraints on renormalization group flows.
- E.g. $SU(2)$ pure gauge theory at $\theta = \pi$ has a mixed anomaly between CP and the $\mathbb{Z}_2$ one-form center symmetry. The low energy phase cannot be trivially gapped with a non-degenerate ground state. (Contrast with the expectation at $\theta = 0$.) [Gaiotto-Kapustin-Komargodski-Seiberg 2017]

Higher-groups

- Higher-group symmetry: mixture of higher-form symmetries of different degrees [Kapustin-Thorngren 2013, Tachikawa 2017, Cordova-Dumitrescu-Intriligator 2018-2020, Benini-Cordova-Hsin 2018,…]. Analogous to group extensions.
- Higher-groups exist in many quantum systems in diverse dimensions: 2+1d Chern-Simons matter theories, 3+1d gauge theories, 5+1d supersymmetric theories…
- Dynamical consequences. E.g. Constraints on the 3+1d axion-Yang-Mills theory [Hidaka-Nitta-Yokokura 2020-2021, Brennan-Cordova 2020].

Subsystem Symmetry

Generalized global symmetries

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible symmetry
Codimension in spacetime	1	> 1	> 1	≥ 1
Topological	yes	yes	not completely but conserved in time	yes
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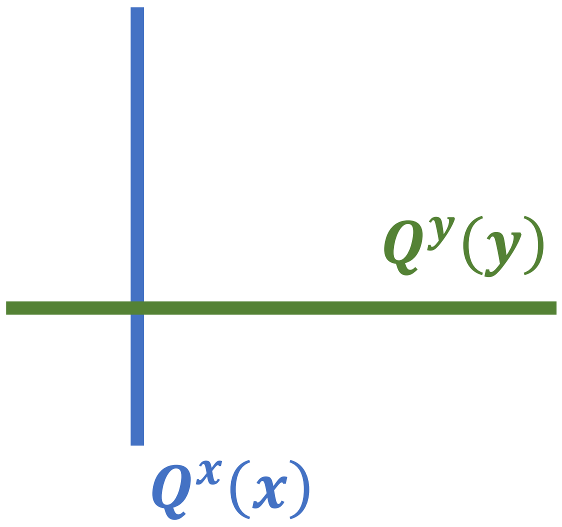
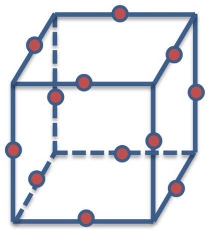
Subsystem symmetry

- There are many interesting lattice models, such as fractons, exhibiting subsystem symmetries.
- The subsystem symmetry charges are supported on certain higher-codimensional manifolds L in space (E.g. straight lines on a plane) [..., Paramekanti-Balent-Fisher 2002, ...]. They depend NOT only on the topology of the manifolds.
- The number of subsystem symmetry charges generally depends on the number of lattice points.
- Low energy observables are sensitive to short distance details: UV/IR mixing [Seiberg-Shao 2020, Gorantla-Lam-Seiberg-SHS 2021].



Fractons



- Fractons [Chamon '05, Haah '11, Vijay-Haah-Fu '16…] are a large class of 3+1d gapped lattice spin models with many peculiar features. Large ground state degeneracy $\sim 2^{\#L}$, where L is the number of lattice sites in every direction, reflecting UV/IR mixing.
- Many fracton models have subsystem symmetries, in which case the peculiarities are universally captured by the symmetries and their anomalies [Seiberg-SHS '20, Burnell-Devakul-Gorantla-Lam-SHS '21] – Generalized Landau paradigm.
- Fractons do **not** admit a conventional continuum field theory limit. Progress in extending the framework of QFT to incorporate these new phases of matter. Higher-rank gauge theory of subsystem symmetries [Pretko 2016x2, Slagle-Kim '17, Bulmash-Barkeshli '18, Ma-Hermele-Chen '18, You-Devakul-Sondhi-Burnell '19, Seiberg-Shao '20, Gorantla-Lam-Seiberg-SHS '20, Qi-Radzihovsky-Hermele '20, Slagle '21, Geng-Kachru-Karch-Nally-Rayhaun '21, Slagle-Hsin '21, Luo-Spieler-Sun-Karch '22…].

Compact Lifshitz theory

- There are even weirder global symmetries in other exotic models motivated by condensed matter systems [Haah 2011, Yoshida 2013, Ma et al. 2020,…].
- Compact Lifshitz theory [Henley 1996, Moessner-Sondhi-Fradkin 2001, Vishwanath-Balents-Senthil 2003,…, Lake-Hermele-Senthil 2022]: (Here ∇^2 is the spatial Laplacian)

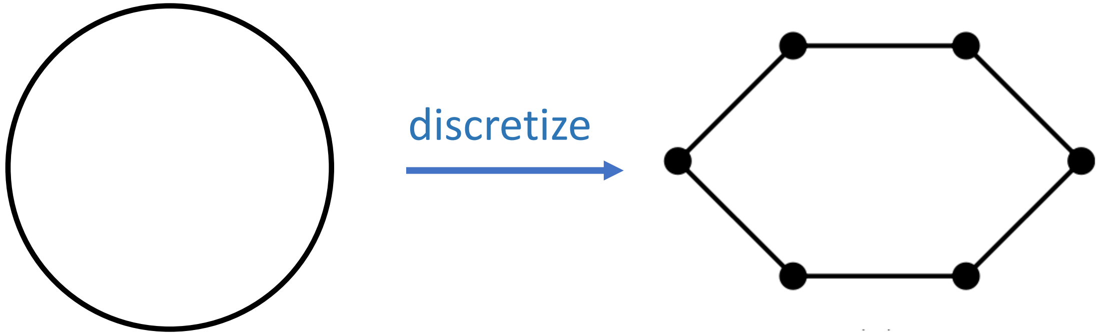
$$\mathcal{L} = \frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\nabla^2 \phi)^2 \ , \qquad \phi \sim \phi + 2\pi$$

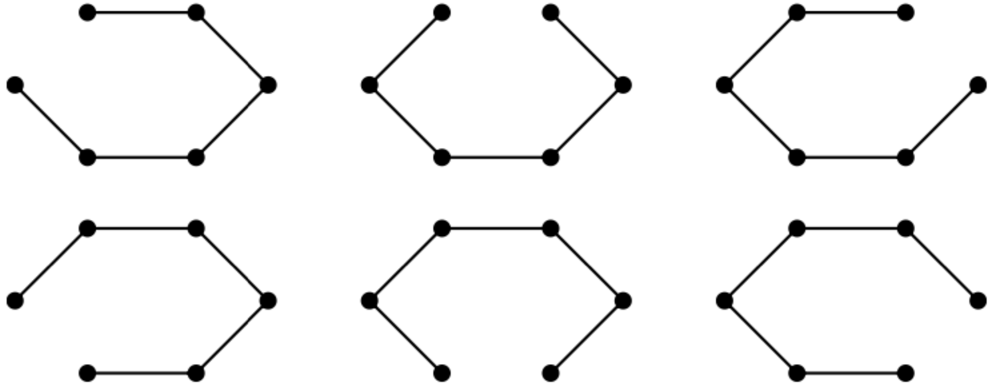
- This continuum theory is ambiguous [Gorantla-Lam-Seiberg-SHS 2022]. For example, on a torus of length ℓ, $\phi = 2\pi n x/\ell$ is a zero-energy solution for all integers n, hence infinite ground state degeneracy. Requires lattice regularization.

GSD = Complexity

[Gorantla-Lam-SHS 2022]

- We discretize the space by a lattice graph.
- In a natural regularization of the theory, the ground state degeneracy (GSD) equals the number of spanning trees of the spatial graph, which is a common measure of complexity in graph theory.
- UV/IR mixing: the GSD, which is an IR observable, equals the complexity of the UV discretization of space.
- New QFT? What is QFT?





Non-invertible Symmetries

Generalized global symmetries

Properties of symmetry op.	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible symmetry
Codimension in spacetime	1	> 1	> 1	≥ 1
Topological	yes	yes	not completely but conserved in time	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	category $\mathcal{D} \times \mathcal{D}^\dagger \neq 1$

“What’s done cannot be undone.”

- Conventionally, a global symmetry is implemented by a unitary operator acting on the Hilbert space [Wigner 1931].
- In particular, conventional symmetry transformations have inverses and can be undone.
- In recent years, we saw rapid developments of a novel kind of global symmetries: non-invertible global symmetry [Bhardwaj-Tachikawa 2017, Chang-Lin-SHS-Wang-Yin 2018, Komargodski-Ohmori-Roumpedakis-Seifnashri 2020, … Choi-Cordova-Hsin-Lam-SHS 2021, Kaidi-Ohmori-Zheng 2021,…].
- It is not implemented by a unitary operator and does not have an inverse. Yet it is an RG-flow invariant and leads to matching conditions.
- Lattice realization [Feiguin et al. 2006, Aasen-Fendley-Mong 2016, 2020, Koide-Nagoya-Yamaguchi 2021, …]. Also known as the algebraic higher symmetry in CMT [Ji-Wen 2019, Kong-Lan-Wen-Zhang-Zheng 2020].

Non-invertible symmetry in QED

- I will discuss one such non-invertible symmetry that exists in Nature – 3+1d QED in the massless limit:

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + i\overline{\Psi}(\partial_\mu - iA_\mu)\gamma^\mu\Psi$$

- It is commonly stated that the classical $U(1)_A$ axial symmetry

$$\Psi \to \exp\left(\frac{i\alpha}{2}\gamma_5\right)\Psi \quad , \qquad \alpha \sim \alpha + 2\pi$$

  is broken by the ABJ anomaly quantum mechanically [Adler 1969, Bell-Jackiw 1969].

- Recently, it was realized that the continuous, invertible $U(1)_A$ axial symmetry is not completely broken, but it is resurrected as a discrete, non-invertible global symmetry labeled by the rational numbers [Choi-Lam-SHS 2022, Cordova-Ohmori 2022].

QED

- The axial current $j_\mu^A = \overline{\Psi}\gamma_5\gamma_\mu\Psi$ obeys the anomalous conservation equation
$$d \star j^A = \frac{1}{4\pi^2} F \wedge F$$
- Naively, we can define the symmetry operator
$$U_\alpha(M) = \exp(\frac{i\alpha}{2}\oint_M \star j^A)$$
- However, it is **not** conserved (topological) because of the anomalous conservation equation.
- Adler defined a gauge non-invariant current which is formally conserved, $d \star \hat{j}^A = 0$,
$$\star \hat{j}^A \equiv \star j^A - \frac{1}{4\pi^2} AdA$$
- But the symmetry operator is **not** gauge invariant on a general three-manifold M
$$\widehat{U}_\alpha(M) = \exp[\frac{i\alpha}{2}\oint_M (\star j^A - \frac{1}{4\pi^2} AdA)]$$

Dilemma

Operator	Gauge-invariant?	Conserved (topological)?
$U_\alpha(M) = \exp(\frac{i\alpha}{2}\oint_M \star j^A)$	✓	✗
$\hat{U}_\alpha(M) = \exp[\frac{i\alpha}{2}\oint_M (\star j^A - \frac{1}{4\pi^2} AdA)]$	✗	✓

Rational angles

- Let us be less ambitious, and assume the axial rotation angle is a fraction:

$$\alpha = \frac{2\pi}{N}$$

$$\hat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{2N} \star j^A - \frac{i}{4\pi N} AdA)]$$

- The operator $\hat{U}_{\frac{2\pi}{N}}(M)$ is still not gauge invariant because of the fractional Chern-Simons term.

Fractional quantum Hall state

$$\text{“}-\frac{i}{4\pi N}\oint_M AdA\text{”}$$

- In condensed matter physics, this action is commonly used to describe the $\nu = 1/N$ fractional quantum Hall effect (FQHE) in 2+1d.
- It is however not gauge invariant. Fortunately, there is a well-known fix to this issue.
- The more precise, gauge invariant Lagrangian for the FQHE is

$$\oint_M \left(\frac{iN}{4\pi} ada + \frac{i}{2\pi} adA\right)$$

where a is a dynamical $U(1)$ gauge field living on the 2+1d manifold M.

- The two actions are related by illegally integrating out a to obtain “$a = -\frac{A}{N}$”.

Back to QED

[Choi-Lam-SHS 2022, Cordova-Ohmori 2022]

- Motivated by the discussion of FQHE in 2+1d, we define a new operator $\mathcal{D}_{1/N}(M)$ in 3+1d QED: (Easy to generalize to $\mathcal{D}_{p/N}$ with coprime p, N)

$$\hat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{2N} \star j^A - \frac{i}{4\pi N} AdA)]$$

a: auxiliary field on M
A: bulk gauge field

$$\downarrow$$

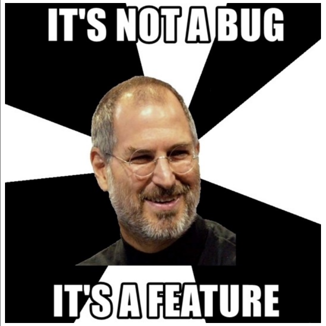
$$\mathcal{D}_{1/N}(M) \equiv \exp[\oint_M (\frac{2\pi i}{2N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

- Easy to generalize to $\mathcal{D}_{p/N}$.
- The new operator is gauge-invariant *and* conserved (topological). The price we pay is that it **NOT** invertible.

$$\mathcal{D}_{1/N} \times \mathcal{D}_{1/N}^{\dagger} = \exp[\oint_M (\frac{iN}{4\pi} ada - \frac{iN}{4\pi} \bar{a}d\bar{a} + \frac{i}{2\pi}(a - \bar{a})dA)] \neq 1$$

Non-invertible global symmetry

Operator	Gauge-invariant?	Conserved (topological)?	Invertible?
$U_\alpha(M) = \exp(\frac{i\alpha}{2}\oint_M \star j^A)$	✓	✗	N/A
$\widehat{U}_\alpha(M) = \exp[\frac{i\alpha}{2}\oint_M (\star j^A - \frac{1}{4\pi^2} AdA)]$	✗	✓	✓
$\mathcal{D}_{\frac{1}{N}}(M) =$ $\exp[\oint_M (\frac{2\pi i}{2N}\star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$	✓	✓	✗



Electron mass

- Naturalness ['t Hooft 1980]: Impose a global symmetry group G. The Lagrangian should include all G-invariant terms with coefficients of order one with no fine-tuning.
- Massless QED Lagrangian: $\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\Psi}(\partial_\mu - iA_\mu)\gamma^\mu \Psi$
- The electron mass term $m\bar{\Psi}\Psi$ violates the non-invertible global symmetry.
- Therefore, electron is naturally massless in QED because of the non-invertible global symmetry.
- See [Cordova-Ohmori 2022] for more discussions on naturalness in axion physics.

't Hooft Naturalness

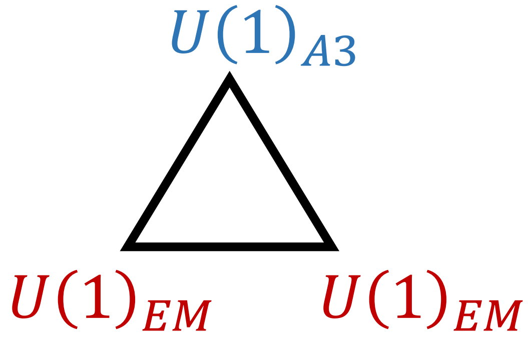
III2. NATURALNESS IN QUANTUM ELECTRODYNAMICS

Quantum Electrodynamics as a renormalizable model of electrons (and muons if desired) and photons is an example of a "natural" field theory. The parameters α, m_e (and m_μ) may be small independently. In particular m_e (and m_μ) are very small at large μ. The relevant symmetry here is chiral symmetry, for the electron and the muon separately. We need not be concerned about the Adler-Bell-Jackiw anomaly here because the photon field being Abelian cannot acquire non-trivial topological winding numbers[4].

't Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking* (1980)

QCD and pion decay

[Choi-Lam-SHS 2022]



- Below the electroweak scale, the massless QCD Lagrangian for the up and down quarks has an axial global symmetry (corresponding to π^0)

$$U(1)_{A3}\colon \begin{pmatrix} u \\ d \end{pmatrix} \to \exp(i\alpha\gamma_5\sigma_3)\begin{pmatrix} u \\ d \end{pmatrix}$$

- The ABJ anomaly with the electromagnetic $U(1)_{EM}$ gauge symmetry turns $U(1)_{A3}$ into an infinite non-invertible global symmetry $\mathcal{D}_{p/N}$.
- The $\pi^0 F \wedge F$ coupling in the IR pion Lagrangian is necessary to match this non-invertible **global** symmetry in QCD.
- To put it in the maximally offensive way, the neutral pion decays $\pi^0 \to \gamma\gamma$ because of the non-invertible global symmetry.

Conclusion

- We have discussed three generalizations of global symmetries, higher form symmetries, subsystem symmetries, and non-invertible symmetries. Many other generalizations.
- This more general perspective of global symmetry unifies many known phenomena into a coherent framework.
  - Generalized global symmetries and their anomalies provide an invariant characterization of many topological phases of matter such as fractons. [John’s talk]
- More importantly, they lead to new dynamical consequences that are otherwise obscured.
  - Generalizations of the ‘t Hooft anomaly matching condition lead to nontrivial constraints on renormalization group flows. Naturalness problems?
- **New** symmetries in **new** and **old** QFTs, including our Nature!

Generalized global symmetries

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Thank you for listening!

Non-invertible symmetry

Snowmass meeting
at KITP in Feb.

1960s	1988-	2010	2017-	2021	2022 Mar	2022 May	
Conserved charges in integrable systems	Non-inv lines in 1+1d RCFT 2+1d TQFT	Non-inv surfaces in 2+1d TQFT	Non-inv op as generalized sym Constraints on RG in 1+1d	3+1d gauge theories	Non-inv sym from higher-form sym	Non-inv sym in Nature	???

Above I mostly focus on codim-1 non-inv op.
Many many other developments not listed.

Non-invertible symmetries in our dimensions

- Non-invertible symmetries exist in many familiar lattice and continuum quantum systems: 3+1d Maxwell theory, Yang-Mills theory, QED, QCD, $\mathcal{N} = 4$ super Yang-Mills…
- Some of them (but not all) arise from invertible symmetries via gauging.
- They have generalized anomalies which lead to nontrivial constraints on renormalization group flows [Choi-Cordova-Hsin-Lam-SHS 2021,2022, Kaidi-Ohmori-Zheng 2021].
- They have been used to construct new QFTs via twisted compactifications [Kaidi-Zafrir-Zheng 2022].
- In quantum gravity, the no global symmetry conjecture is argued to be generalized to the absence of invertible *and* non-invertible global symmetries [Rudelius-SHS 2020, Heidenreich-McNamara-Montero-Reece-Rudelius-Valenzuela 2021, McNamara 2021].
- Other hidden symmetries in the Standard Model?